A CAT algorithm for 2-polyominoes

June 21th, 2017 • Journées MDSC • Nice, France.

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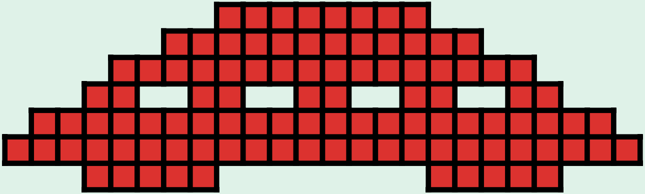
Polyomino

Any finite 4-connected subset of $\mathbb{Z}^2$



Polyominoes

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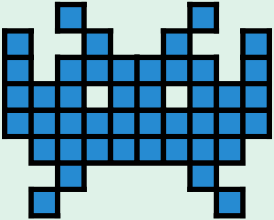

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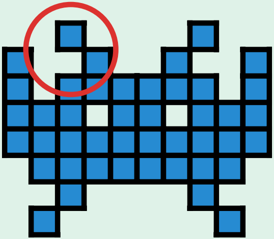


Polyominoes: history and motivations

Introduction: Golomb (1954)
First extensive study: ???
Successively: too many!

Polyominoes are widely used in many fields:

- Enumerative combinatorics
- Bijective combinatorics
- Two dimensional language theory
- Tilings
- Discrete tomography

Polyominoes: the open questions

Enumeration

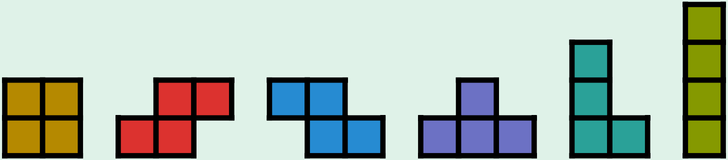
Can polyominoes of a given surface n be efficiently enumerated?

Closed formula

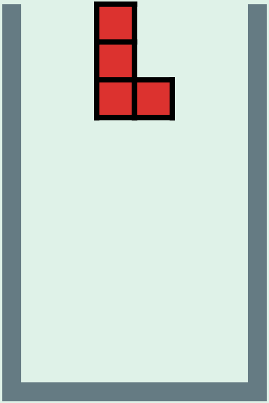
Is there a closed formula for the number of polyominoes of a given surface n?

Tetris

All variants of tetrominos fall one-by-one from top to bottom.



Classic Tetris



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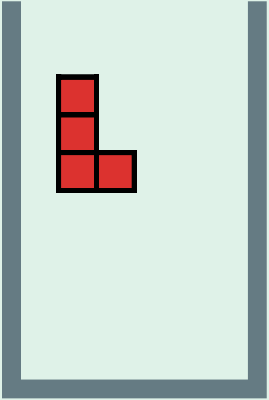

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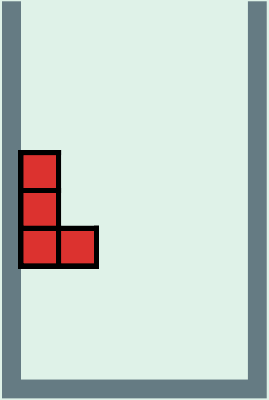

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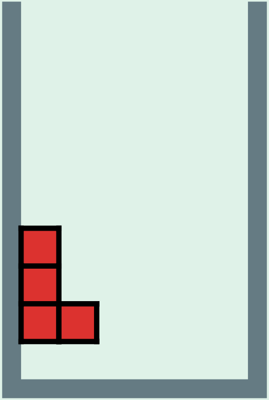

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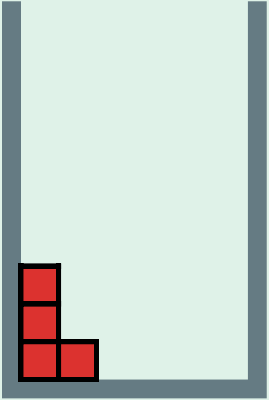

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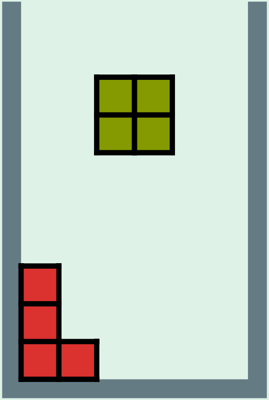

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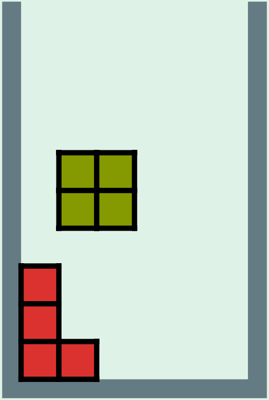

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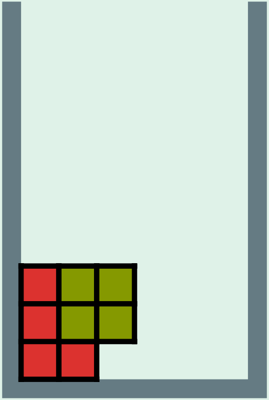

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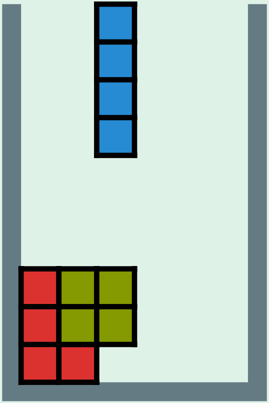

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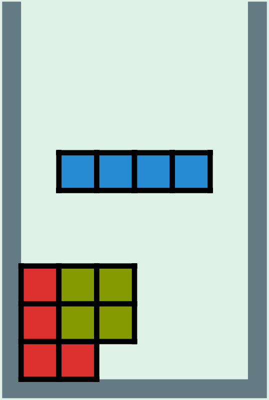

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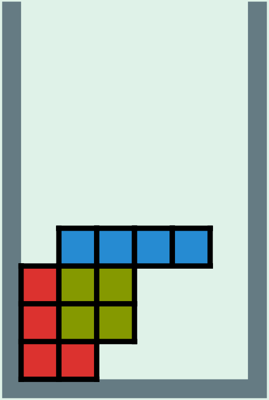

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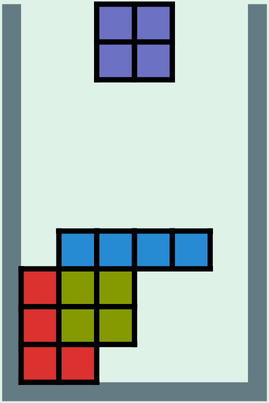

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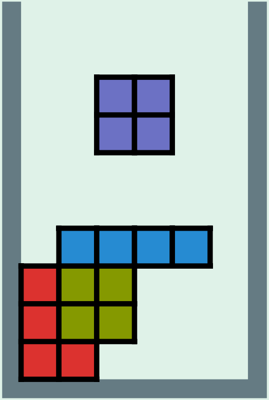

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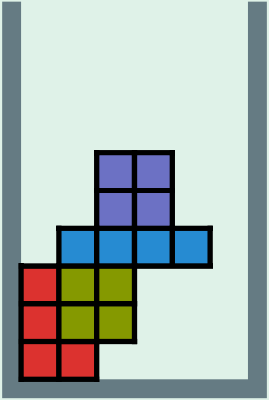

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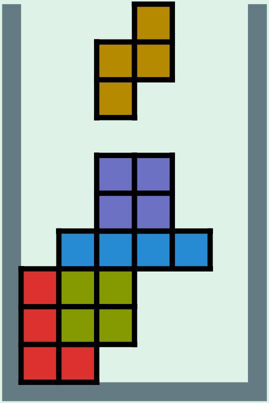

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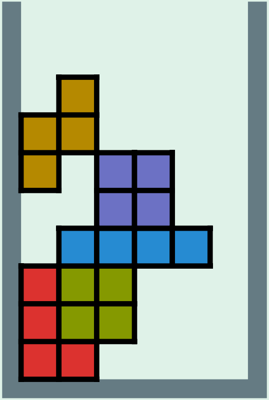

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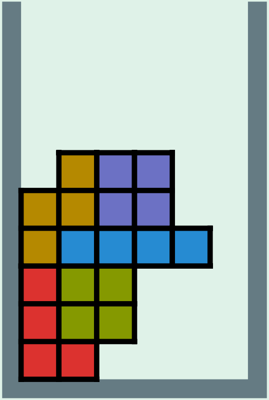

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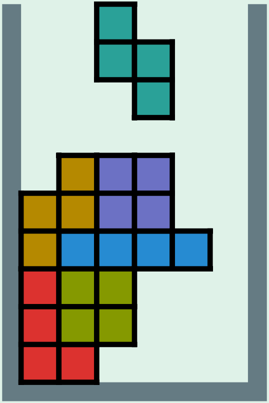

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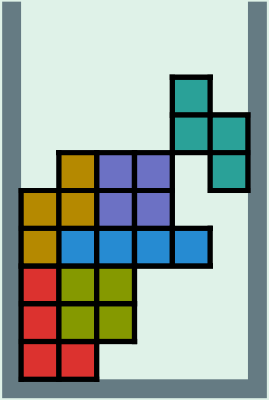

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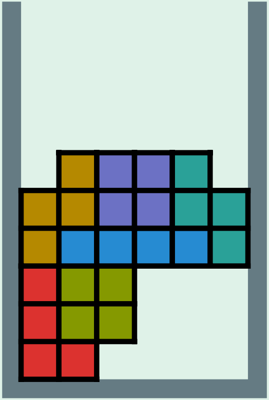

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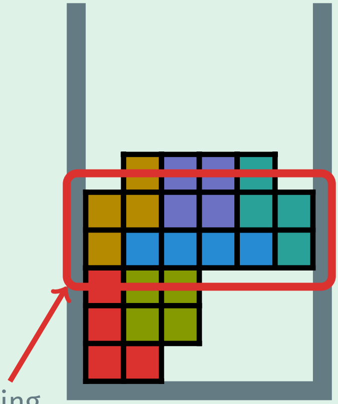
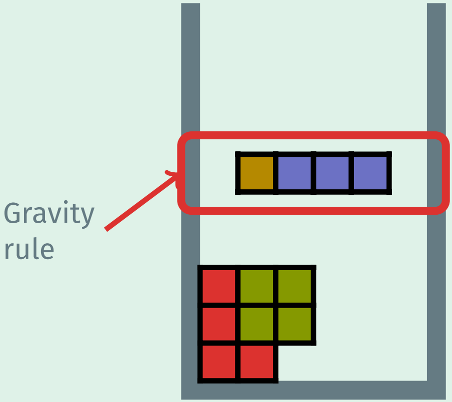


Cancelling rule

Classic Tetris



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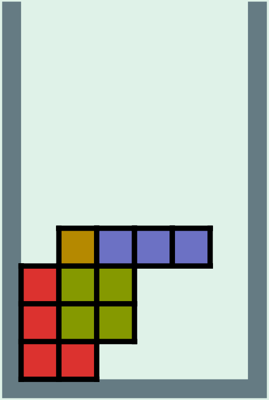

Bad Tetris and bad players

Pieces: any bar of length $\leq k$ (bad Tetris)

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Pieces:	any bar of length $\leq k$	(bad Tetris)
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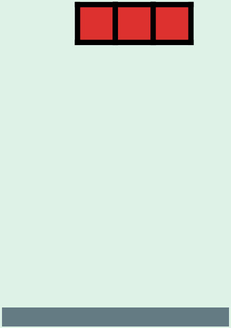
Surface:	fix $s \in \mathbb{N}$	(bad player)
-Rules:	No cancelling	(bad Tetris)
+Rules:	New pieces and old ones have to be 4-connected	(bad player)
-Rules:	No more left and right walls	(bad Tetris)

Bad Tetris (and bad player)



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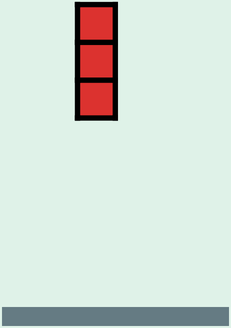

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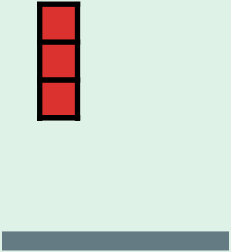

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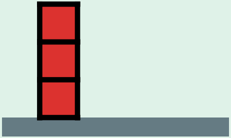

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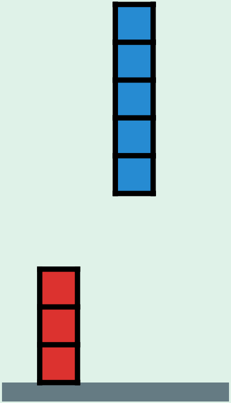

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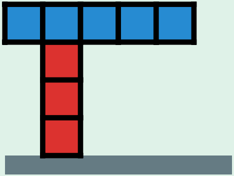

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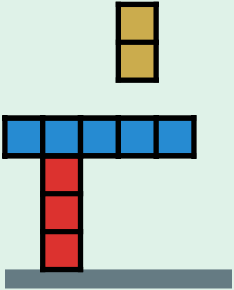

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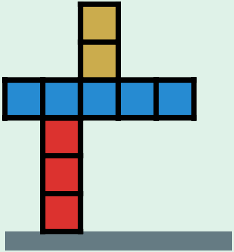

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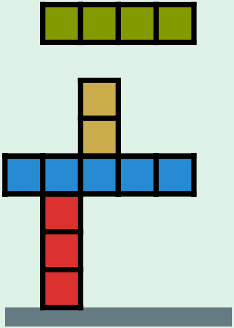

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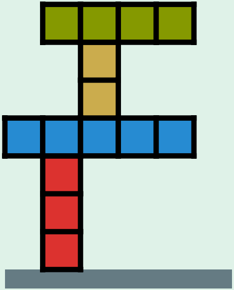


Polyomino representation

Polyomino

Ordered sequence of columns.

Column

Sequence of integers $\{\alpha_1, x_1, \alpha_2, x_2, \ldots, \alpha_k, x_k\}$
st.

α_1	=	vertical displacement wrt previous column
x_i	=	segment length
α_i	=	hole length ($i > 1$)

Some notation

Given $P = \{C_1, C_2, \ldots, C_i, \ldots C_k\}$ with
$C_1 = \{\alpha_1, x_1, \alpha_2, x_2, \ldots, \alpha_k, x_k\}$ and $C_1 = \{\alpha'_1, x'_1, \alpha'_2, x'_2, \ldots, \alpha'_l, x'_l\}$

$P_i = C_i$ (i-th column of P)

$P_{\leq i} = \{C_1, \ldots, C_i\}$

$C_i \Box C_{i-1}$ iff no segment of C_i falls completely in a hole of C_{i-1}

Comparing columns

$C_1 = \{\alpha_1, x_1, \alpha_2, x_2, \ldots, \alpha_k, x_k\}$

$C_2 = \{\alpha'_1, x'_1, \alpha'_2, x'_2, \ldots, \alpha'_l, x'_l\}$

$C_1 \leq C_2$ if one of the following holds

1. $\sum_{i=1}^{k} x_i < \sum_{j=1}^{l} x'_i$
2. $\sum_{i=1}^{k} x_i = \sum_{j=1}^{l} x'_i$ but $\alpha_1 > \alpha'_1$
3. $\sum_{i=1}^{k} x_i = \sum_{j=1}^{l} x'_i$ and $\alpha_1 = \alpha'_1$ but $\exists e$ s.t.
   3.1 $x_i = x'_i$ and $\alpha_i = \alpha'_i$ for $1 < i < e$
   3.2 $x_e > x'_e$ if e is even
   3.3 $\alpha_e < \alpha'_e$ if e is odd

Comparing Polyominoes

$P = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_i, \ldots \mathbf{c}_k\}$

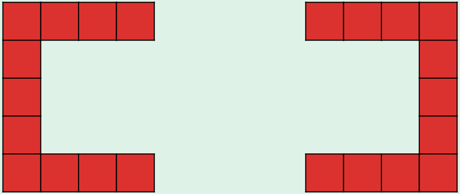
$Q = \{\mathbf{c}'_1, \mathbf{c}'_2, \ldots, \mathbf{c}'_i, \ldots \mathbf{c}'_l\}$

$P \leq Q$ iff $\exists i$ s.t. $\mathbf{c}_i < \mathbf{c}'_i$ and $\mathbf{c}_h = \mathbf{c}'_h$ for $h < i$.

Prefix-closed polyominos

Prefix-closed polyomino

$P = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_i, \ldots \mathbf{c}_k\}$ is prefix-closed iff for all $i \in \{1, \ldots, k\}$, $P_{\leq i}$ is a polyomino.



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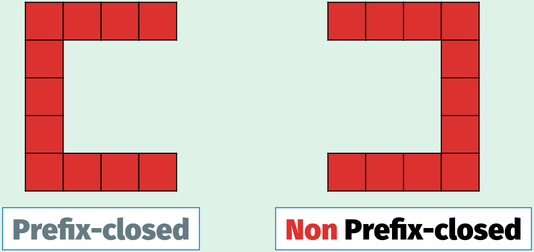
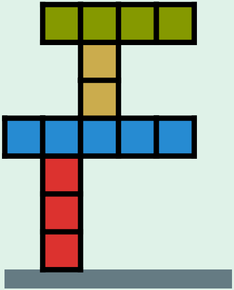


Bad Tetris + bad player = Prefix-closed Polyominoes



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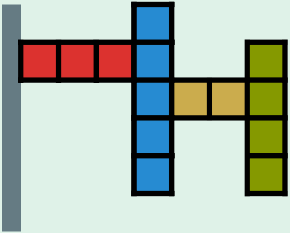

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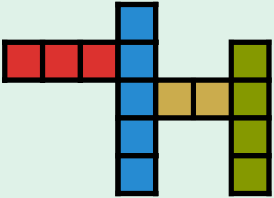
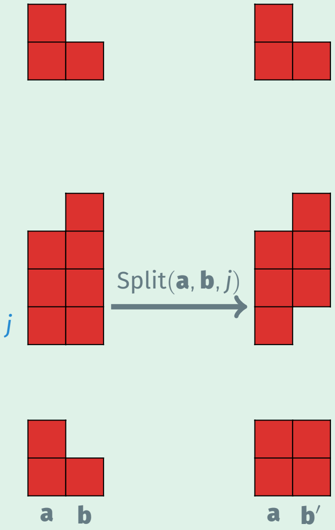


The result

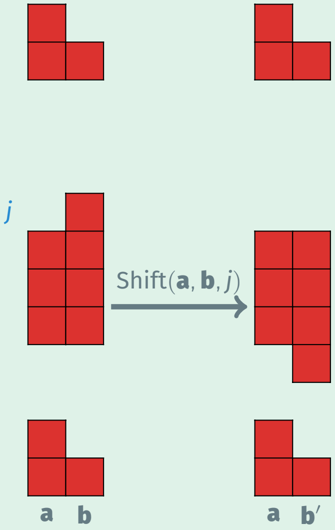
Theorem

There exist a CAT algorithm for exhaustive generation of PCPol(n).

Generating columns: split move



Generating columns: shift move



Generating columns: the grand ancestor

Grand ancestor

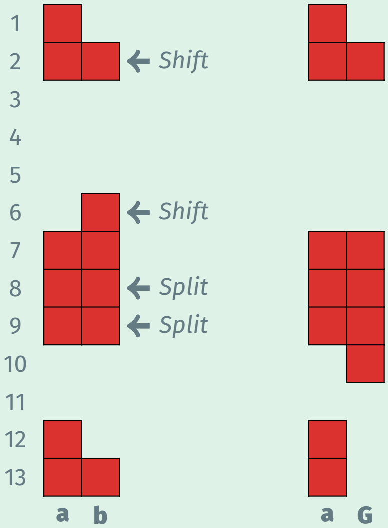
$C_1 = \{\alpha_1, x_1, \alpha_2, x_2, \dots, \alpha_k, x_k\}$

$C_2 = \{\alpha'_1, x'_1, \alpha'_2, x'_2, \dots, \alpha'_l, x'_l\}$

$G(C_2, C_1) =$ column which is

1. the smallest $\square$-compat. with C_1
2. identical to C_2 up to $j = \max\{M(C_2)\}$
3. admits a move at j
4. has same surface as C_2

Generating columns: the grand ancestor



$$M(b) = \{2, 6, 8, 9\}$$

Generating columns: the dynamical system

$C(\mathbf{a}, r)$ = set of all columns of area r and $\square$-compatible with $\mathbf{a}$

$\mathbf{b}' \xrightarrow{j} \mathbf{b}$ if $\mathbf{b}' = \mathrm{Split}(\mathbf{a}, \mathbf{b}, j)$ or $\mathbf{b}' = \mathrm{Shift}(\mathbf{a}, \mathbf{b}, j)$

$$f_{\mathbf{a},r}(\mathbf{b}) = \begin{cases} \mathbf{b}' & \exists \mathbf{b}', \mathrm{GA}(\mathbf{b}, \mathbf{a}) \xrightarrow{j} \mathbf{b}',\ j = \max M(\mathbf{b}) \\ (\alpha_1 - h(\mathbf{a}) + 1, r) & \text{otherwise} \end{cases}$$

Generating columns: the dynamical system (cont.)

Lemma

Fix an integer r and a column

$$\mathbf{a} = (\alpha_1, x_1, \alpha_2, x_2, \dots, \alpha_k, x_k)$$

then for all $\mathbf{b}, \mathbf{d} \in C(\mathbf{a}, r)$ it holds:

1. $f_{\mathbf{a},r}(\mathbf{b}) \square \mathbf{a}$
2. $\mathbf{b} < \mathbf{d}$ implies $f_{\mathbf{a},r}(\mathbf{b}) < f_{\mathbf{a},r}(\mathbf{d})$;
3. $f^n_{\mathbf{a},r}(\mathbf{b}) < f^{n+1}_{\mathbf{a},r}(\mathbf{b})$ for $\mathbf{b} \neq (\alpha_1 - h(\mathbf{a}) + 1, r)$;
4. $\bigcup_{n\in\mathbb{N}} f^n_{\mathbf{a},r}((r-1, r)) = C(\mathbf{a}, r)$.

Generating PCPol(n): the algorithm

```
PROCEDURE PCPOLGEN(n)
for r := 1 to n − 1 do
    P_1 := (0, r); COLGEN(2, n − r);
end for
P := (0, n); OUTPUT(P);
```

Generating PCPol(n): the algorithm (columns)

PROCEDURE COLGEN(i, r)
for d := 1 to r **do**
    INITCOLUMN(i); {i.e. P_i:= $((d), d-1)$}
    INIT(S, i); {init stack}
    if $d < r$ **then** COLGEN($i+1, r-d$); **else** OUTPUT(P); **endif**
    while not ISEMPTY(S) **do**
        j:=GRAN(i, S); {restore the grand ancestor}
        MOVE(j, S); {execute a move at j and update S}
        if $d < r$ **then** COLGEN($i+1, r-d$); **else** OUTPUT(P);
        endif
    end while
end for

Generating PCPol(n): the algorithm (columns)

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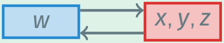

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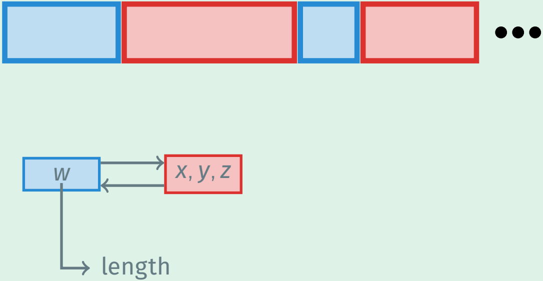

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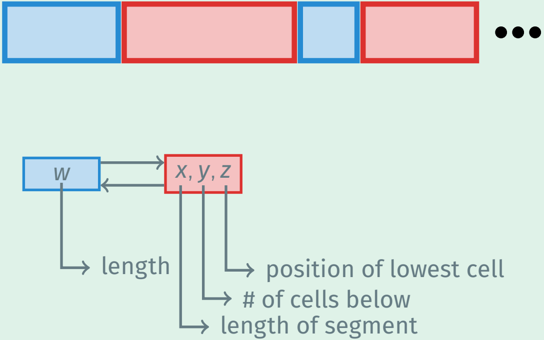

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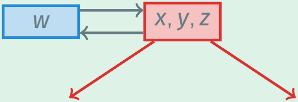

ptr to smallest compat. segment of prev. column

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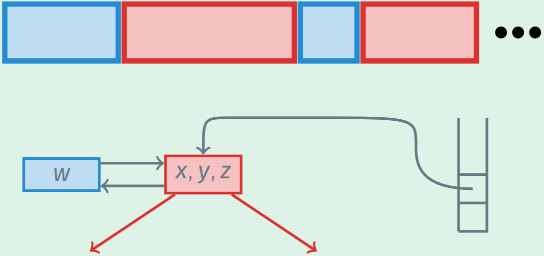

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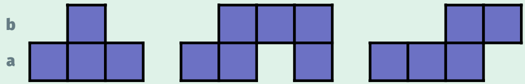
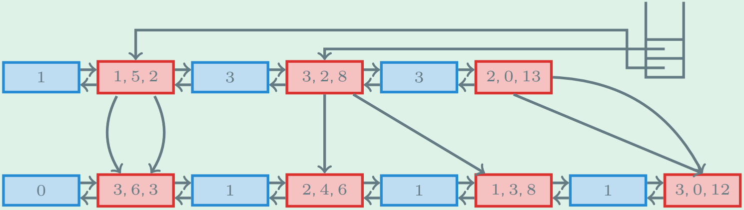
ptr to largest compat. segment of prev. column

Generating PCPol(n): an example





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- Please, go meta!!!

Thank you.