Secret Sharing through Cellular Automata

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One-Dimensional Cellular Automata (CA)

Definition

One-dimensional cellular automaton: triple $\langle n, r, f \rangle$ where $n \in \mathbb{N}$ is the number of cells arranged on a one-dimensional array, $r \in \mathbb{N}$ is the radius and $f : \{0,1\}^{2r+1} \to \{0,1\}$ is the local rule.

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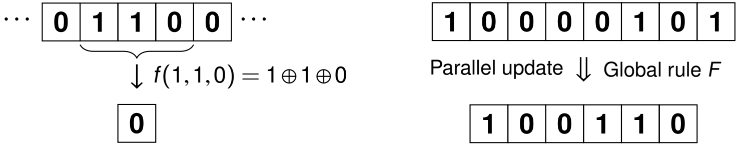
Example: $n = 8$, $r = 1$, $f(s_{i-1}, s_i, s_{i+1}) = s_{i-1} \oplus s_i \oplus s_{i+1}$ (Rule 150)



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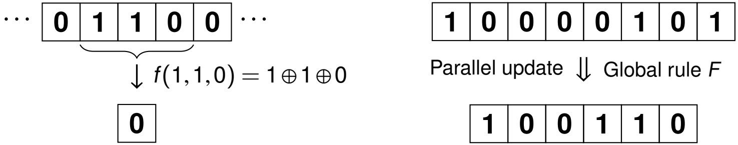

Remark: No boundary conditions ⇒ The array "shrinks"

Secret Sharing Schemes (SSS)

- Secret sharing scheme: a procedure enabling a dealer to share a secret S among a set $\mathcal{P}$ of n players
- In (k, n) threshold schemes, at least k players out of n are required to recover S

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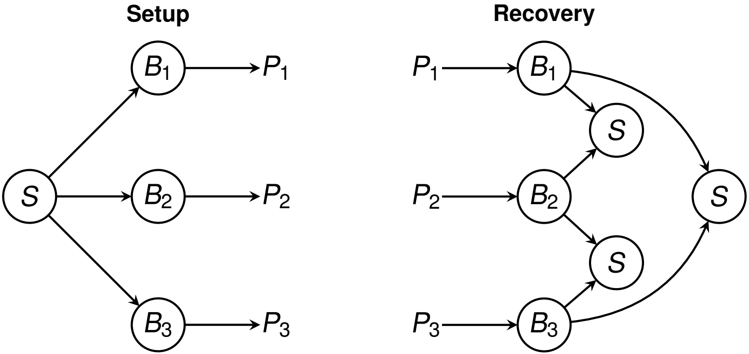
Example: $(2,3)$–scheme



Bipermutive Rules

- Rule $f : \{0,1\}^{2r+1} \rightarrow \{0,1\}$ is called bipermutive if there exists $g : \{0,1\}^{2r-1} \rightarrow \{0,1\}$ such that:

$$f(x_1, x_2, \cdots, x_{2r}, x_{2r+1}) = x_1 \oplus g(x_2, \cdots, x_{2r}) \oplus x_{2r+1}$$

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- A preimage $p \in \{0,1\}^{m+2r}$ of $c \in \{0,1\}^m$ is uniquely determined by a block of $2r$ cells

$p =$ | ? | ? | ? | ? | 0 | 1 | ? | ? |

$c =$ | 1 | 0 | 0 | 1 | 1 | 0 |

(a) Initialization

$p =$ | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |

$c =$ | 1 | 0 | 0 | 1 | 1 | 0 |

(b) Complete preimage

Figure : Example with bipermutive rule 150

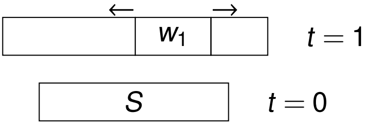
Basic (n, n) Secret Sharing Scheme - Setup Phase

1. The *dealer* D sets the secret S as an m-bit configuration of a CA, and selects a bipermutive rule of radius r such that $2r|m$

S	$t = 0$

Basic (n, n) Secret Sharing Scheme - Setup Phase

2. D evolves the CA backwards for $T = m(n-1)/2r$ iterations, randomly choosing an initial $2r$-bit block at each step



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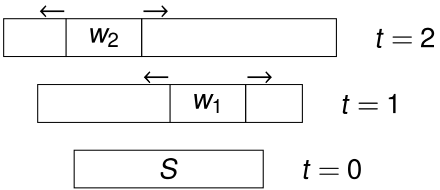
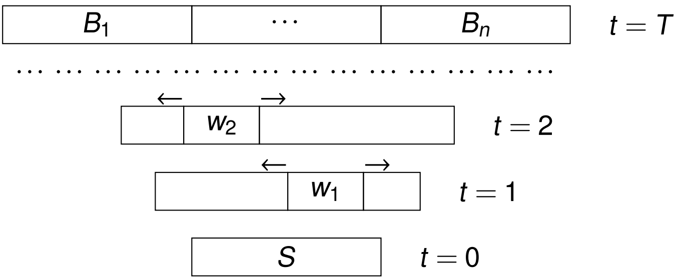


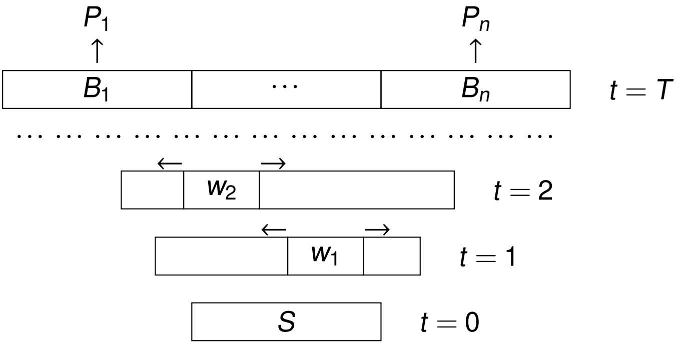
Basic (n, n) Secret Sharing Scheme - Setup Phase

3. After $T = m(n-1)/2r$ iterations, the dealer splits the resulting preimage in n blocks of m bits



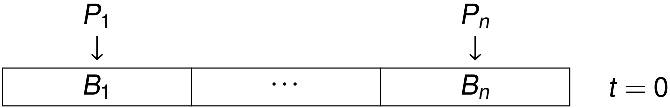
Basic (n, n) Secret Sharing Scheme - Setup Phase

4. D securely sends one block to each player and publishes the bipermutive rule used



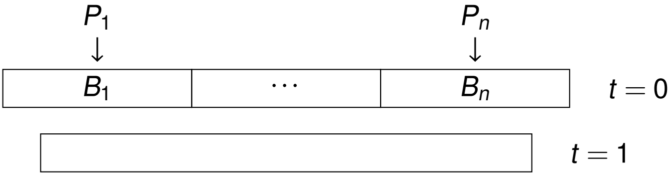
Basic (n, n) Secret Sharing Scheme - Recovery Phase

1. The n players pool their shares in the correct order to get the complete preimage of the CA



Basic (n, n) Secret Sharing Scheme - Recovery Phase

2. The players evolve the CA forward, using the local rule published by the dealer



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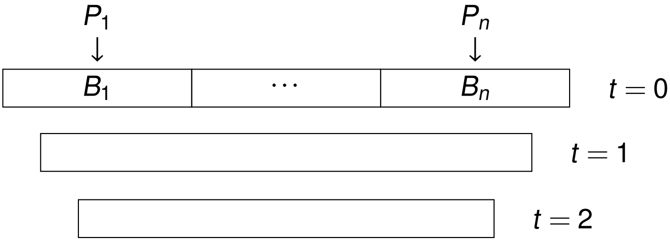
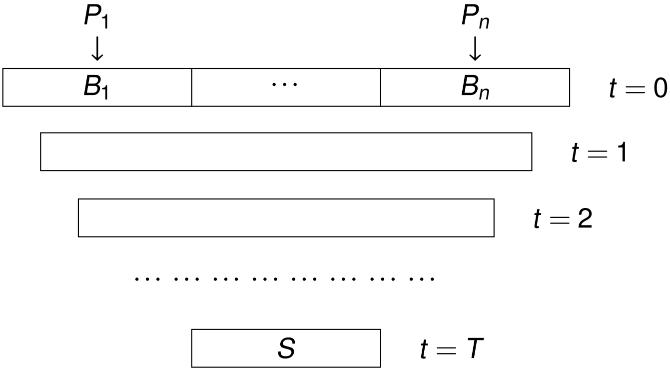


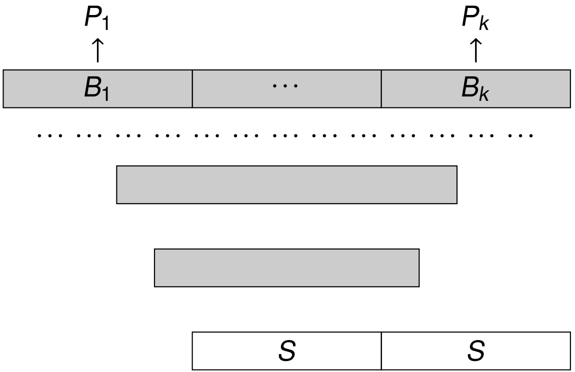
Basic (n, n) Secret Sharing Scheme - Recovery Phase

3. The configuration obtained after $T = m(n-1)/2r$ iterations is the secret S.



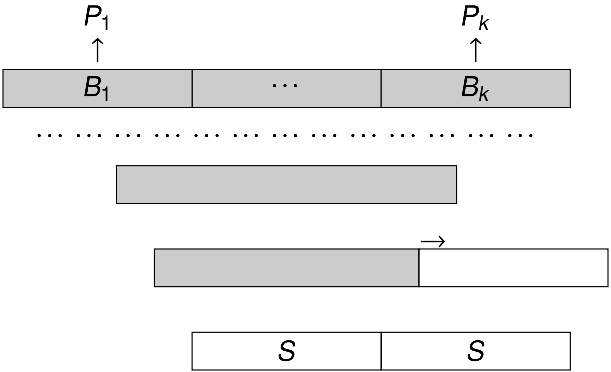
Secret Juxtaposition (1/4)

1. Append a copy of the secret S to the right of the final CA image



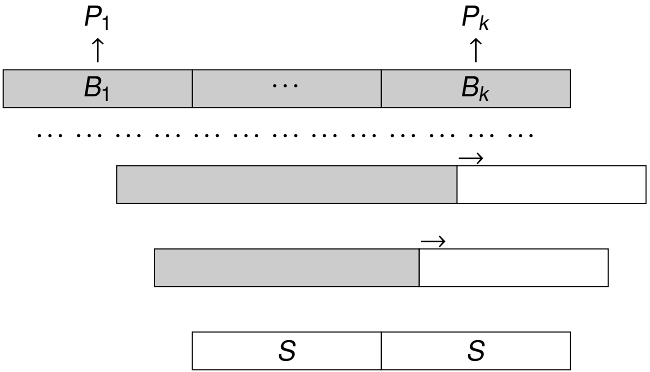
Secret Juxtaposition (2/4)

2. Update the preimages by completing them rightwards (note that it is not necessary to pick extra random bits)



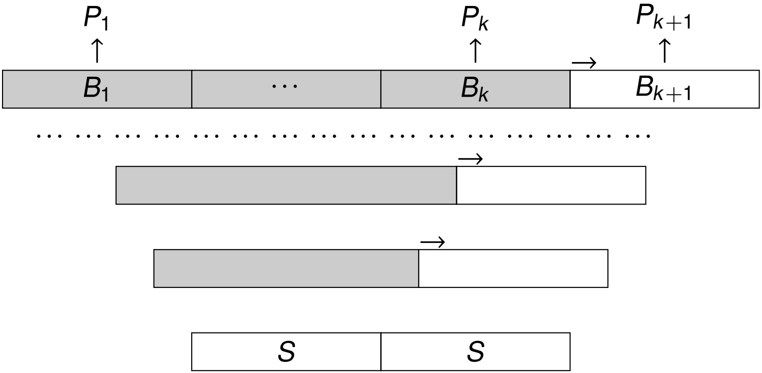
Secret Juxtaposition (3/4)

2. Update the preimages by completing them rightwards (note that it is not necessary to pick extra random bits)



Secret Juxtaposition (4/4)

3. The last preimage contains an additional block for the new player. The sets $\{P_1, \cdots, P_k\}$ and $\{P_2, \cdots, P_{k+1}\}$ can recover S



Access Structure of the Scheme

- (k, n)-sequential threshold: at least k consecutive shares are necessary to recover the secret
- By continuing to append copies of the secret, the shares will eventually repeat $\Rightarrow$ cyclic access structure

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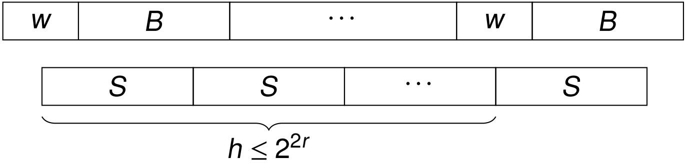

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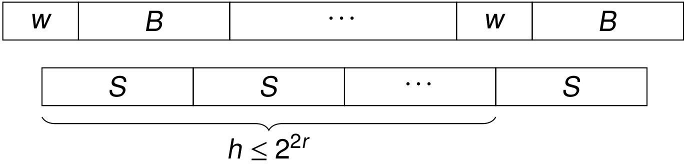

What about real threshold schemes with CA?

A Different Angle: Latin Squares

Definition

A *Latin square* of order N is a $N \times N$ matrix L from such that every row and every column are permutations of $[N] = \{1, \cdots, N\}$

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

Orthogonal Latin Squares

Definition

Two Latin squares L_1 and L_2 of order n are *orthogonal* if their superposition yields all the pairs $(x, y) \in [N] \times [N]$.

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

(a) L_1

1	4	2	3
3	2	4	1
4	1	3	2
2	3	4	1

(b) L_2

1,1	3,4	4,2	2,3
4,3	2,2	1,4	3,1
2,4	4,1	3,3	1,2
3,2	1,3	2,1	4,4

(c) (L_1, L_2)

A set of n pairwise orthogonal Latin squares is denoted as n-MOLS

$(2,n)$-Schemes through n-MOLS

1. The dealer D chooses a row $S \in \{1, \cdots, N\}$ as the secret

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

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	4	3	2	1
→	2	1	4	3
	3	4	1	2

	1	2	3	4
	3	4	1	2
→	4	3	2	1
	2	1	4	3

	1	2	3	4
	2	1	4	3
→	3	4	1	2
	4	3	2	1

Example: $(2, 3)$-scheme, $S = 3$

$(2,n)$-Schemes through n-MOLS

2. D randomly selects a column $j \in \{1, \cdots, N\}$

		↓		
	1	2	3	4
	4	3	2	1
→	2	1	4	3
	3	4	1	2

		↓		
	1	2	3	4
	3	4	1	2
→	4	3	2	1
	2	1	4	3

		↓		
	1	2	3	4
	2	1	4	3
→	3	4	1	2
	4	3	2	1

Example: $S = 3$, $j = 2$

$(2, n)$-Schemes through n-MOLS

3. The value of $L_i(S, j)$ for $i \in [n]$ is the share of P_i

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Example: $(2, 3)$-scheme, $S = 3$, $j = 2$, $B_1 = 1$, $B_2 = 3$, $B_3 = 4$

4. Since L_i, L_k are orthogonal, (B_i, B_k) uniquely identify (S, j)

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

1	2	3	4
2	1	4	3
3	4	1	2
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Example: $(2,3)$-scheme, $B_1 = 1$, $B_2 = 3 \Rightarrow (3,2)$

$(2, n)$-Schemes through n-MOLS

4. Since L_i, L_k are orthogonal, (B_i, B_k) uniquely identify (S, j)

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

$\rightarrow$

	$\downarrow$		
1	2	3	4
3	4	1	2
4	(3)	2	1
2	1	4	3

$\rightarrow$

	$\downarrow$		
1	2	3	4
2	1	4	3
3	(4)	1	2
4	3	2	1

Example: $(2,3)$-scheme, $B_2 = 3$, $B_3 = 4 \Rightarrow (3,2)$

$(2,n)$-Schemes through n-MOLS

4. Since L_i, L_k are orthogonal, (B_i, B_k) uniquely identify (S, j)

1	2	3	4
4	3	2	1
2	(1)	4	3
3	4	1	2

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

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3	(4)	1	2
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Example: $(2,3)$-scheme, $B_1 = 1$, $B_3 = 4 \Rightarrow (3,2)$

Latin Squares through Bipermutive CA

- Problem reduction: determine which CA induce orthogonal Latin squares

Lemma

Let $\langle 2m, r, t, f \rangle$ be a bipermutive CA with $2r|m$. Then, the CA generates a Latin square of order $N = 2^m$

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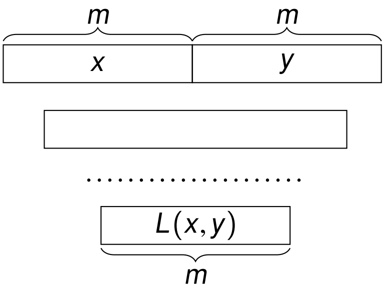

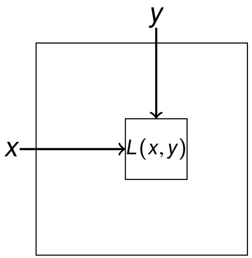

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0000 00	0010 11	0001 01	0011 10
1000 10	1010 01	1001 11	1011 00
0100 11	0110 00	0101 10	0111 01
1100 01	1110 10	1101 00	1111 11

(a) Rule 150

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) L_{150}

$$00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$$

Linear CA

- Local rule: linear combination of the neighborhood cells

$$f(x_0, \cdots, x_{2r}) = a_0 x_0 \oplus \cdots \oplus a_{2r} x_{2r} \ \ , \ a_i \in \mathbb{F}_2$$

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- Associated polynomial:

$$f \mapsto P_f(X) = a_0 + a_1 X + \cdots + a_{2r} X^{2r}$$

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- Associated polynomial:

$$f \mapsto P_f(X) = a_0 + a_1 X + \cdots + a_{2r} X^{2r}$$

- Global rule: $m \times (m + 2r)$ $2r$-diagonal transition matrix

$$M_F = \begin{pmatrix} a_0 & \cdots & a_{2r} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_0 & \cdots & a_{2r} & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_0 & \cdots & a_{2r} \end{pmatrix}$$

$$x = (x_0, \cdots, x_{n-1}) \mapsto M_F x^\top$$

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$$x = (x_0, \cdots, x_{n-1}) \mapsto M_F x^\top$$

- $a_0, a_{2r} \neq 0 \Rightarrow f$ bipermutive

Orthogonal Latin Squares by Linear CA

Theorem

The Latin squares induced by $\langle 2m, r, t, f\rangle$ *and* $\langle 2m, r, t, g\rangle$ *are orthogonal if and only if* $gcd(P_f(X), P_g(X)) = 1$

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1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(a) Rule 150

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(b) Rule 90

1,1	4,2	3,3	2,4
2,2	3,1	4,4	1,3
4,3	1,4	2,1	3,2
3,4	2,3	1,2	4,1

(c) Superposition

Figure : $P_{150}(X) = 1 + X + X^2$, $P_{90}(X) = 1 + X^2$ (coprime)

Conclusions and Perspectives

- Recap:
  - A single bipermutive CA can be used to implement a (k, n) sequential threshold scheme
  - A set of n linear CA with coprime rules gives rise to a set of n MOLS (and thus to a $(2, n)$-threshold scheme)
- Future developments:
  - Count (and build!) pairs of coprime polynomials
  - Generalise to higher threshold (using orthogonal hypercubes)

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