On the conjecture about deterministic reversal bounded counter machines ($\mathcal{L}_{\mathrm{DFCM}} \subsetneq \mathrm{RCM}$)

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Abstract

We study the relationship between RCM and some classes of languages defined by deterministic reversal bounded counter machines.

We prove that:

- $\mathcal{L}_{\mathrm{DFCM}(1,0,1)} \subsetneq \mathrm{RCM}$
- $\mathcal{L}^{\mathrm{fin}}_{\mathrm{DFCM}(1,0,1)} \subsetneq \mathrm{RCM}$
- $\mathcal{L}^{\mathrm{quot}}_{\mathrm{DFCM}(1,0,1)} \subsetneq \mathrm{RCM}$

We exploit a simulation technique that provides a possible basis to prove the conjecture $\mathcal{L}_{\mathrm{DFCM}} \subsetneq \mathrm{RCM}$ (this conjecture implies that the generating function of a language in $\mathcal{L}_{\mathrm{DFCM}}$ is holonomic).

Generating functions and languages

The (ordinary) generating function of $L \subseteq \Sigma^\star$ is

$$\phi(x)_L = \sum_{w \in L} x^{|w|} = \sum_{n \geq 0} a_n x^n$$

It is well known (Chomsky-Schützenberger 1963) that

- L is regular $\implies$ $\phi(x)_L$ is rational
- L is unambiguos c.f. $\implies$ $\phi(x)_L$ is algebraic

Flajolet's methodology: use analytic properties of $\phi(x)_L$ to decide properties of L

L c.f. and $\phi(x)_L$ transcendental $\implies$ L inherently ambiguous

The class RCM

We consider classes of languages specified by three elements

- a language $L \in \mathcal{L}$, $L \subseteq \Gamma^\star$
- a system of constraints C
- a morphism $\mu : \Gamma^\star \mapsto \Sigma^\star$

Definition (linear constraint)

A linear constraint on the number of occurrences of symbols of $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_h\}$ in $w \in \Gamma^\star$ is

$$\sum_{i=1}^{h} c_i |w|_{\gamma_i} \bigtriangleup c_{h+1}, \quad \text{with } c_i \in \mathbb{Z}, \bigtriangleup \in \{<, \leq, =, \neq, \geq, >\}.$$

A system of linear constraints C is either a linear constraint, or $C_1 \vee C_2$ or $C_1 \wedge C_2$ or $\neg C_1$, where C_1, C_2 are systems of linear constraints.

The class RCM

Notation

- $[C] = \{w \in \Gamma^\star | w \text{ satisfies } C\}$
- $\langle L, C, \mu \rangle = \mu(L \cap [C]) \subseteq \Sigma^\star$

Definition (RCM – Castiglione, Massazza - 2016)

$\text{RCM} = \{\langle R, C, \mu \rangle \subseteq \Sigma^\star\}$ where

- $R \subseteq \Gamma^\star$ is regular
- C is a system of linear constraints on Γ
- $\mu : \Gamma^\star \mapsto \Sigma^\star$ is length preserving and injective on $R \cap [C]$.

RCM: properties

Definition (Bernstein '71)

A function $f(x) : \mathbb{C} \mapsto \mathbb{C}$ is *holonomic* if and only if there exist $d \in \mathbb{N}$ and $p_i(x) \in \mathbb{C}[x]$, $0 \leq i \leq d$, with $p_d(x) \neq 0$, such that

$$\sum_{i=0}^{d} p_i(x) \frac{d^i}{dx^i} f(x) = 0.$$

Main properties of RCM (Castiglione, Massazza - 2016):

- $L \in \text{RCM} \implies \phi_L(x)$ is holonomic
- RCM is closed under union and intersection
- the following decision problems are decidable for RCM: equivalence, inclusion, disjointness, emptiness, universe

RCM: relationship with other classes of languages

$$\mathcal{L}_{\mathrm{LPA}} \subsetneq \mathrm{RCM}, \quad \mathrm{RCM} \not\subseteq \mathcal{L}_{\mathrm{DFCM}}, \quad \mathrm{RCM} \subsetneq \mathcal{L}_{\mathrm{NFCM}}$$

- $\mathcal{L}_{\mathrm{LPA}}$ is the class of languages recognized by Parikh automata on letters (Cadilhac, Finkel, McKenzie - 2012)
- $\mathcal{L}_{\mathrm{DFCM}}$ (resp., $\mathcal{L}_{\mathrm{NFCM}}$) is the class of languages recognized by deterministic (resp., non deterministic) one-way 1-reversal bounded counter machines (Ibarra - 1978)

Conjecture (Castiglione, Massazza - 2016)

$$\mathcal{L}_{\mathrm{DFCM}} \subsetneq \mathrm{RCM}$$

Deterministic 1-reversal bounded k-counter machines

Definition (DFCM(k,0,1))

$M \in$ DFCM(k,0,1) is a DFA with one-way input tape and equipped with k-counters that can switch from increasing to decreasing mode at most once

W.l.o.g. we suppose that M always terminates and that words are accepted with all the counters equal to 0

Our result: a particular case of the conjecture

Theorem

$$\mathcal{L}_{\mathrm{DFCM}(1,0,1)} \subsetneq \mathrm{RCM}$$

Corollaries of $\mathcal{L}_{\mathrm{DFCM}(1,0,1)} \subsetneq \mathrm{RCM}$

Corollary ($\mathcal{L}^{\mathrm{fin}}_{\mathrm{DFCM}(1,0,1)} \subsetneq \mathrm{RCM}$)

Let $L_1, L_2 \in \mathcal{L}_{\mathrm{DFCM}(1,0,1)}$. Then, one has

$$L_1 \cap L_2, L_1 \cup L_2 \in \mathrm{RCM}$$

Furthermore, by exploiting a result of Eremondi, Ibarra and McQuillan (2015) one has

Corollary ($\mathcal{L}^{\mathrm{quot}}_{\mathrm{DFCM}(1,0,1)} \subseteq \mathrm{RCM}$)

Let $L \in \mathcal{L}_{\mathrm{DFCM}(1,0,1)}$ and $L_1, L_2 \in \mathcal{L}_{\mathrm{NPCM}}$. Then, one has

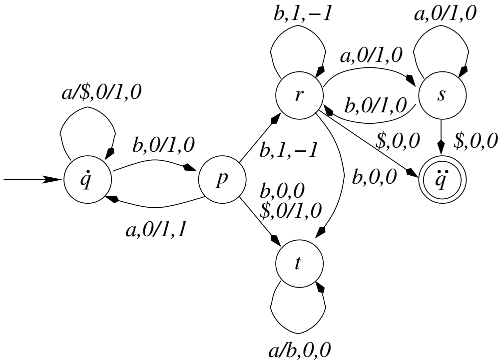
$$(L_1^{-1}L)L_2^{-1}, L_1^{-1}(LL_2^{-1}) \in \mathrm{RCM}$$

Remark: NPCM = non deterministic PDA augmented with a fixed number of counters

DFCM$(1,0,1)$: an example

Consider

$$\sharp\text{IsolB} = \{xbby \mid x, y \in \{a,b\}^\star, |x|_{bb} = 0 \wedge |bby|_{bb} = |x|_b\}$$



Simulating $M \in \mathrm{DFCM}(1, 0, 1)$: the idea

During the computation, the global state of the counter of $M \in \mathrm{DFCM}(1, 0, 1)$ belongs to $\{0, 1, 2, 3\}$:

0 (never incremented) $Q_0 \subseteq Q$

1 (incremented but not decremented) $Q_1 \subseteq Q$

2 (incremented and decremented) $Q_2 \subseteq Q$

3 (incremented, decremented, equal to 0) $Q_3 \subseteq Q$

To simulate M it is sufficient to know

- the transition function δ_M:
- the current state of M
- the global state of the counter ($0 \to 1 \to 2 \to 3$)
- the current value of the counter (an integer)

Simulating $M \in \text{DFCM}(1,0,1)$

Given $M \in \text{DFCM}(1,0,1)$ find A', C, μ s.t.

$$\langle L(A'), C, \mu \rangle = L(M)$$

The DFA A' on Γ has the following characteristics:

- the global state of the counter of M is directly encoded in the states of A', $|Q'| \leq 4|Q|$
$$Q' = \{q_\alpha | 0 \leq \alpha \leq 3, q \in Q_\alpha\}$$
- A' uses weighted symbols to simulate the counter. If M has a (sequence of) transition(s) consuming σ that
  1. increments the counter by k then add to Γ a symbol σ_k
  2. decrements the counter by k then add to Γ two symbols σ_{-k} (used to guess that the counter remains greater than zero) and σ'_{-k} (used to guess that the counter is equal to k)

Simulating $M \in \text{DFCM}(1, 0, 1)$: transition function $\delta_{A'}$

$\delta_{A'}$: the transition function of A'

If M is in p with the counter in state i and

$$(p, \sigma x\$, c) \Rightarrow (q, x\$, c + k)$$

then define

$$\delta_{A'}(p_i, \sigma_k) = q_j$$

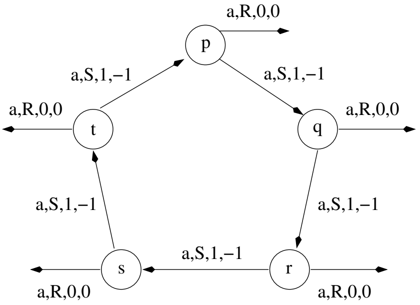
for a suitable $j \geq i$ (depending on the state transition of the counter), and also

$$\delta_{A'}(p_i, \sigma'_k) = q_3$$

if $k < 0$

Simulating $M \in \mathrm{DFCM}(1,0,1)$: negative cycles

Problem: M may have negative cycles that reset the counter



- A transition from p to q (resp., $r, s, \ldots$) corresponds to $c \bmod 5 = 1$ (resp., $c \bmod 5 = 2$, $c \bmod 5 = 3, \ldots$)
- At most one negative cycle can be traversed!
- If M enters a negative cycle it can only check the modulus of the counter

Negative cycles: the counter

Fact

If M admits negative cycles the exact value c of the counter is not needed! Just remember $e = c \bmod \pi$ where $\pi = \prod_{d \in D} |d|$, D is the set of the weights of the negative cycles of M. Indeed, for any $d \in D$ one has

$$c \bmod d = e \bmod d$$

Solution:

- If M has negative cycles then the states of A' should incorporate the modulus of the counter, i.e. Q' is

$$\{p_\alpha(0), q_\beta(i), | \alpha \in \{0,3\}, \beta \in \{1,2\}, p \in Q_\alpha, q \in Q_\beta, 0 \leq i < \pi\}$$

- add to Γ a symbol σ'' if in M there is a cycle on σ

Negative cycles: transition function $\delta_{A'}$

$\delta_{A'}$: the transition function of A' (in case of negative cycle)

If M is in p (belonging to a negative cycle of weight d), the counter is in the state $i \in \{1, 2\}$ and

$$(p, \sigma x\$, c) \Rightarrow (q, x\$, 0)$$

with $c \bmod d = r$ (r uniquely depends on p and q), then for all j s.t. $j \bmod d = r$ define

$$\delta_{A'}(p_i(j), \sigma'') = q_3(0)$$

Simulating $M \in \mathrm{DFCM}(1,0,1)$: the constraints and μ

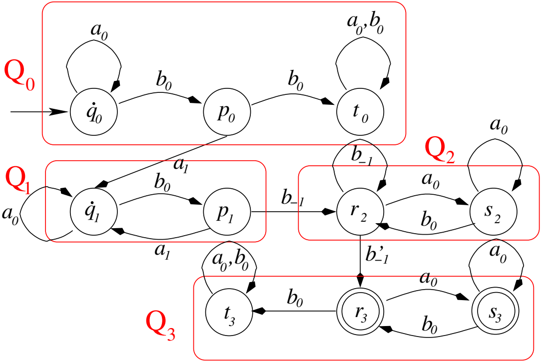
Remark: if $w \in L(A')$ then w consists only of symbols of weight 0 or w contains exactly one guess symbol (either σ'' or σ'_i)

The system of linear constraints is $C_1 \vee C_2 \vee C_3$ where

$$
\begin{aligned}
C_1 &: \quad \sum_{\sigma'_i} |w|_{\sigma'_i} = 0 \wedge \sum_{\sigma''} |w|_{\sigma''} = 0 \wedge \sum_{i \neq 0} |w|_{\sigma_i} = 0, \\
C_2 &: \quad \sum_{\sigma'_i} |w|_{\sigma'_i} = 1 \wedge \sum_{\sigma''} |w|_{\sigma''} = 0 \wedge \sum_{\sigma_i, \sigma'_i} (i|w|_{\sigma_i} + i|w|_{\sigma'_i}) = 0, \\
C_3 &: \quad \sum_{\sigma'_i} |w|_{\sigma'_i} = 0 \wedge \sum_{\sigma''} |w|_{\sigma''} = 1 \wedge \sum_{\sigma_i} i|w|_{\sigma_i} > 0.
\end{aligned}
$$

What about the morphism μ? it simply ignores the indices

The RCM specification of ♯IsolB



- $\mu(a_0) = \mu(a_1) = a \quad \mu(b_{-1}) = \mu(b_0) = \mu(b_1) = \mu(b'_{-1}) = b$
- $C = |w|_{b'_{-1}} = 1 \wedge |w|_{a_1} - |w|_{b_{-1}} - |w|_{b'_{-1}} = 0$

Conclusions

Proving that $\mathcal{L} \subseteq \mathrm{RCM}$ provides us with interesting results regarding many decision problems and counting for $\mathcal{L}$

Future works

- Generalize the technique to k counters (easy if M has no negative cycles), that is, prove $\mathcal{L}_{\mathrm{DFCM}} \subsetneq \mathrm{RCM}$
- Develop automatic tools for RCM (e.g. closure properties, counting)
- Analyze the relationship between RCM and other classes of languages
- Extend the Flajolet's methodology to RCM

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